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NOTE ON THE EQUATIONAL DEFINABILITY OF ADDITION IN RINGS

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Boolean rings and Boolean algebras, through historically and conceptually different, were shown by M.H. Stone to be equationally interdefinable. Indeed, in a Boolean ring, addition is definable in terms of multiplication and the successor operation (Boolean complementation) $x\hat{} = x+1$: $x+y = \{(xy\hat{})\hat{}(x\hat{}y)\hat{}\}\hat{} = (x\hat{}y\hat{})\hat{}(xy)\hat{}$.

In Theorem 1 of [2], H.G. Moore and A. Yaqub proved that this type of equational definability of addition also holds for rings satisfying the identity $x^n = x^{n+k}$ in which the idempotents are in the center. More generally, in Theorem 2 of [2] it was shown that this equational definability of addition still holds when the identity $x^n = x^{n+k}$ above is replaced by the identity $x^n = x^{n+1}f(x)$, $f(t) \in \mathbb{Z}[t]$.

However, the following proposition will show that the hypotheses assumed in Theorems 1 and 2 of [2] are equivalent.

Proposition ([1, Proposition]). If R is a ring with identity, then the following are equivalent:

1) R is normal (every idempotent in R is central) and there exists a positive integer n and a polynomial

$f(t) \in \mathbb{Z}[t]$ such that $x^n = x^{n+1}f(x)$ for all $x \in R$.

2) There exists a positive integer n and a polynomial $f(t) \in \mathbb{Z}[t]$ such that $x^n = x^{n+1}f(x)$ and $(xy)^n f(xy)^n = (yx)^n f(yx)^n$ for all $x, y \in R$.

3) There exists a positive integer n and a polynomial $f(t) \in \mathbb{Z}[t]$ such that $(xy)^n = (yx)^{n+1}f(yx)$ for all $x, y \in R$.

4) There exists a positive integer n and a polynomial $f(t) \in \mathbb{Z}[t]$ such that $x^n = x^{n+1}f(x)$ and $(xy)^n = (yx)^n$ for all $x, y \in R$.

5) R is normal and there exist positive integers n, k such that $x^n = x^{n+k}$ for all $x \in R$.

Proof. The equivalence of 3) and 4) is immediate.

1) $\Rightarrow$ 5). Clearly $qR = 0$, where $q = |2^{n+1}f(2) - 2^n|$ (>1). We set $d = \deg f(t)$ (≥ 0), and $k = q^{d+1}$. If x is nilpotent, then we readily obtain $x^{nk} = x^{nk+(nk)!}$ $(=0)$. Next, we consider the case that x is not nilpotent. Evidently, $e = x^n f(x)^n$ is an idempotent with $x^n = x^n e = (xe)^n$. Let $y = xe = ex$. Since $e = y^n f(y)^n$ and $y^n = y^{n+1}f(y)$, $y^* = f(y)e$ is the inverse of y in eRe. Then, it is easy to see that $|\langle y^* \rangle| \leq k$, and that $y^{*\ell} = e$ with some positive integer $\ell < k$. Hence, we obtain $(x^n)^\ell = (y^n)^\ell = (y^\ell)^n = e$, and thus $x^{nk} = x^{nk+n\ell} = x^{nk+(nk)!}$.

5) $\Rightarrow$ 4). It is easy to see that there exists a positive integer m such that $x^m = x^{2m} = x^{m+1}x^{m-1}$ for all $x \in R$. Now, let x, y be arbitrary elements of R. Since $(xy)^m$

and $(yx)^m$ are central idempotents, we have

$$(xy)^m = x(yx)^{m-1}(yx)^m y = (yx)^m (xy)^m,$$

and similarly $(yx)^m = (xy)^m (yx)^m$. Hence, $(xy)^m = (yx)^m$.

4) $\Rightarrow$ 2). Actually, $(xy)^n f(xy)^n = f(xy)^n (yx)^{2n} f(yx)^n = (xy)^{2n} f(xy)^n f(yx)^n = (xy)^n f(yx)^n = (yx)^n f(yx)^n$.

2) $\Rightarrow$ 1). Let e be an arbitrary idempotent of R. By 2), for any unit u of R we have

$$e = e^n = e^{2n} f(e)^n = e^n f(e)^n = (euu^{-1})^n f(euu^{-1})^n$$
$$= (u^{-1}eu)^n f(u^{-1}eu)^n = u^{-1}(e^n f(e)^n)u = u^{-1}eu.$$

Hence, e commutes with all units, and therefore with all nilpotents. This proves that e is central.

We shall conclude this note with giving an elegant proof to Theorem 1 of [2]: Since $q = 2^{n+k} - 2$ is zero in R, we have $x^{\vee} = x - 1 = x + (q-1)$ for any $x \in R$. Let x, y be arbitrary elements of R. By hypothesis, $e = x^{nk}$ and $e' = (x+1)^{nk}$ are central idempotents of R such that $ex^n = x^n$ and $e'(x+1)^n = (x+1)^n$. Without loss of generality, we may assume that n is odd. Since

$$1 - e = (x^n + 1)(1 - e)$$
$$= (x+1)^n (x^{n-1} - x^{n-2} + \ldots - x + 1)(1-e) = e'(1-e),$$

we can easily see that

$$x + y = (ey + x)e + \{e'(y-1) + x + 1\}(1-e)$$
$$= \{(ey+x)e + 1\}[\{e'(y-1) + x + 1\}(1-e) + 1] - 1$$
$$= \{x^{nk+1}(x^{nk-1}y + 1) + 1\} \times$$
$$[(x+1)(x^{nk} - 1)^2\{(x+1)^{nk-1}(y-1) + 1\} + 1] - 1$$
$$= [\{x^{nk+1}(x^{nk-1}y)^{\wedge}\}^{\wedge}\{x^{\wedge}((x^{nk})^{\vee})^2((x^{\wedge})^{nk-1}y^{\vee})^{\wedge}\}^{\wedge}]^{\vee}.$$

References

[1] H. Abu-Khuzam, H. Tominaga and A. Yaqub: Equational definability of addition in rings satisfying polynomial identities, Math. J. Okayama Univ. 22 (1980), 55-57.

[2] H.G. Moore and A. Yaqub: Equational definability of addition in certain rings, Pacific J. Math. 74 (1978), 407-417.